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Computer-Aided Analysis and Design of Networks Containing Commensurate and Noncommensurate Delay Lines

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***Abstract*—Several computer programs are described for the analysis and synthesis of networks containing transmission lines, lumped resistors, voltage sources, and current sources. There are no restrictions on the topology of the networks and degenerate elements can also be included. In the noncommensurate case the transmission lines could have different delays and thus the degree of freedom for each network is doubled. State-space techniques are used to formulate the solution to the problem and the high degree of generality was achieved by using topological methods to derive the state equations. Several examples are given.**

I. Introduction

IN THIS PAPER a general approach is described for the analysis and synthesis of networks containing commensurate and noncommensurate transmission lines. There are no restrictions on the topology; however, for the present work we shall assume that the lines are uniform and nondispersive.

The networks to be considered can be generally represented as shown in Fig. 1. The sections $S_1 \cdots S_s$ contain lumped resistors, voltage sources and current sources. The transmission lines could be either connecting lines between the sections or degenerate lines. There are four types of degenerate lines, these are lines terminated by a short circuit, an ideal voltage source, an open circuit, and an ideal current source. Mutual coupling between the lines could exist and a solution for such circuits has been obtained using techniques similar to those described below. However, the results for networks with mutual coupling will be described in a future publication.

The networks could also be classed as either normal or nonnormal. If it is possible to label each section as either even or odd, such that no two even sections or two odd sections are directly connected by one or more transmission lines and if all the sources are either in odd sections or in even sections, then the network is normal. If such a division is not possible then the network is nonnormal. Both normal and nonnormal networks are treated in this work.

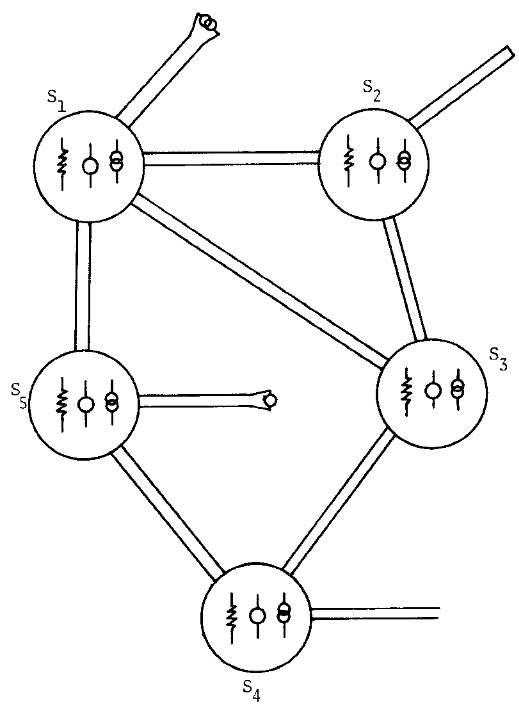

Fig. 1. General network containing transmission lines, resistive elements, and sources.

The most general case will be a network containing noncommensurate lines with mutual coupling between the lines. A computer program devised to deal with these networks will also deal with the more restricted case of uncoupled commensurate lines. However, this approach

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will result in an unnecessary waste of computer time and store. Furthermore, there are a number of techniques that can only be used in the commensurate case and have no equivalent in the noncommensurate case. For example, the roots of the characteristic polynomial or the eigenvalues of the state matrix give the natural frequencies of the commensurate network and can be readily used as the basis of the synthesis procedure. In the noncommensurate case the eigenvalues of the state matrix do not give the natural frequencies of the network and the roots of the characteristic function are infinite. Thus, it is not possible to use either the eigenvalues or the roots as the basis of the synthesis procedure in the noncommensurate case.

The two cases of networks containing commensurate and noncommensurate transmission lines will be treated separately with maximum usage of any common techniques and continuous reference to any similarities.

II. The Basic Matrices

Topological methods were used to formulate the state equations. In this section, we describe these methods and define the basic matrices used to derive the equations.

A. The Forest and Coforest

Given a network containing transmission lines, we proceed to choose a forest and a coforest to describe the topology. First degenerate current and voltage sources are eliminated using the E-shift and I-shift theorems. This step will not affect the four cases of degenerate transmission line terminations mentioned in Section I.

An unconnected graph is then drawn for the network. The edges will either be a resistive edge, a transmission line edge, a degenerate short circuit (or voltage) edge or a degenerate open circuit (or current) edge. These types of edges are shown in Fig. 2. The orientation of the edges should be such that the two edges belonging to the same transmission line are in the same direction as shown in Fig. 2. This is important in the case of non-normal networks but not essential in the case of normal networks.

A forest and a coforest are then chosen for the network such that the following conditions are satisfied:

a) the maximum number of transmission line edges are in the coforest,
b) all the degenerate short circuit or voltage edges are in the coforest and all the degenerate open circuit or current edges are in the forest.

An alternative to Condition a) is that the maximum number of transmission line edges should be in the forest. This will result in an alternative but equally useful form of the state equations. The edges are then numbered starting with the transmission line and degenerate edges in the forest followed by the resistive edges in the forest, the transmission line and degenerate edges in the coforest and finally the resistive edges in the coforest.

B. The Dynamical Transformations Matrix

The dynamical transformations matrix D for the whole network is obtained by assigning its rows to the branches

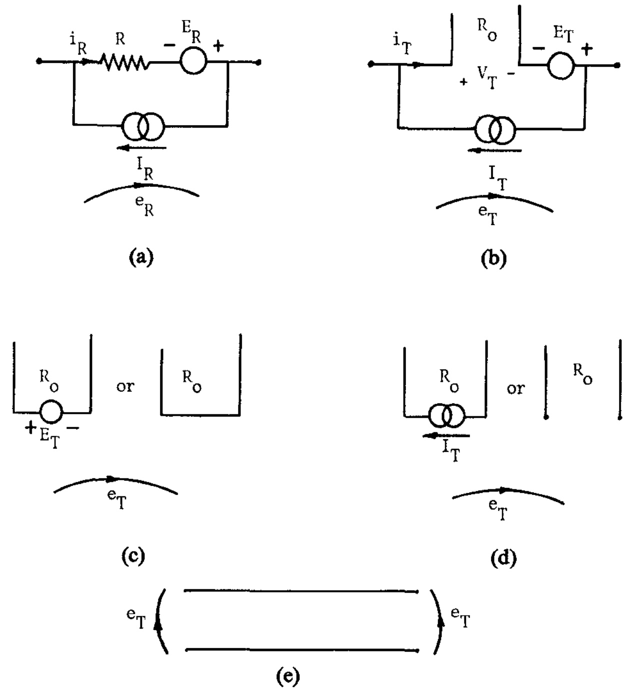


Fig. 2. (a) Resistive edge. (b) Nondegenerate transmission line edge. (c) Degenerate short circuit or voltage edge. (d) Degenerate open circuit or current edge. (e) Direction of edges belonging to the same line.

and its columns to the chords. The columns are tie sets consisting of the assigned chord and as many branches as necessary or alternatively the rows are cut sets consisting of the assigned branch and as many chords as necessary.[1]

When the matrix D is partitioned with respect to the transmission line and resistive chords and branches, the edge currents and voltages are related by

$$\begin{bmatrix} i_{Tf} \\ i_{Rf} \\ \hline v_{Tc} \\ v_{Rc} \end{bmatrix} = \begin{array}{c} \\ \\ m \\ b \end{array} \underset{e\times e}{\begin{bmatrix} 0 & \begin{matrix} D_{TT} & D_{TR} \\ D_{RT} & D_{RR} \end{matrix} \\ \hline \begin{matrix} -D_{TT}^T & -D_{RT}^T \\ -D_{TR}^T & -D_{RR}^T \end{matrix} & 0 \end{bmatrix}} \begin{array}{c} l \\ a \\ \\ \\ \end{array} \cdot \underset{e\times 1}{\begin{bmatrix} v_{Tf} \\ v_{Rf} \\ \hline i_{Tc} \\ i_{Rc} \end{bmatrix}} \begin{array}{c} l \\ a \\ m \\ b \end{array} + \underset{e\times 1}{\begin{bmatrix} I_T \\ I_R \\ \hline E_T \\ E_R \end{bmatrix}} \begin{array}{c} l \\ a \\ m \\ b \end{array} \tag{1}$$

where

$$I_T = I_{Tf} - D_{TT} I_{Tc} - D_{TR} I_{Rc} \tag{2a}$$

$$I_R = I_{Rf} - D_{RT} I_{Tc} - D_{RR} I_{Rc} \tag{2b}$$

$$E_T = E_{Tc} + D_{TT}^T E_{Tf} + D_{RT}^T E_{Rf} \tag{2c}$$

$$E_R = E_{Rc} + D_{TR}^T E_{Tf} + D_{RR}^T E_{Rf}. \tag{2d}$$

[1]Several other names and symbols (F, H, C, $\cdots$, etc.) have been used for this matrix, a standardization is overdue and highly desirable.

The subscripts T and R refer to the transmission line ports and the resistive elements, respectively, and the subscripts f and c refer to the forest and coforest, respectively. In (1) l and m are the number of transmission line edges in the forest and coforest, respectively; $n=(l+m)/2$ is the total number of transmission lines; a and b are the number of resistive edges in the forest and coforest respectively; $d=a+b$ is the total number of resistive edges; and $e=2n+d$ is the total number of edges. When the maximum number of transmission line edges are in the coforest (forest) the matrix $D_{TR}=0$ $(D_{RT}=0)$.

C. The Characteristic Resistance Matrix

This is a matrix of the characteristic resistances of all the transmission lines. When coupling between the lines does not exist, the characteristic resistance matrix R_0 is a diagonal matrix given by

$$R_0=\begin{bmatrix} R_{0f} & 0 \\ 0 & R_{0c} \end{bmatrix}_{2n\times 2n}=G_0^{-1}. \qquad (3)$$

D. The Resistive Elements Matrix

Similarly, all the values of the resistive elements are entered in a diagonal matrix R

$$R=\begin{bmatrix} R_f & 0 \\ 0 & R_c \end{bmatrix}_{d\times d}. \qquad (4)$$

We also define a conductance matrix $\bar{G}$

$$\bar{G}=\left[G_f+D_{RR}R_c^{-1}D_{RR}^T\right]_{a\times a}. \qquad (5)$$

E. The Row Operations Matrix

This matrix rearranges the rows of the state vector such that the two rows belonging to the same transmission line are interchanged. The row operations matrix K is obtained by assigning its columns to the transmission line edges in the order they appear in the state vector, and its rows to the same edges with the two edges belonging to the same transmission line interchanged. An entry of 1 is made at the intersection of the column and row assigned to the same edge and an entry of 0 is made otherwise. It should be noted that the matrix K always has the property that $K=K^T=K^{-1}$.

III. The State Equation for Commensurate Networks

In a commensurate network all the transmission lines have the same delay T. The state vector $[b_f(t):b_c(t)]^T$ of transmission line networks is taken as a vector of the reflected parameters at all the transmission line terminals. With respect to this vector, the state equation can be obtained [1] in terms of the basic matrices given in Section II

$$\begin{bmatrix} b_f(t+T) \\ b_c(t+T) \end{bmatrix}=A\begin{bmatrix} b_f(t) \\ b_c(t) \end{bmatrix}+B_1\begin{bmatrix} I_R(t) \\ E_R(t) \end{bmatrix}+B_2\begin{bmatrix} I_T(t) \\ E_T(t) \end{bmatrix} \qquad (6)$$

where

$$A=B_2\begin{bmatrix} U_l & -D_{TT} \\ -D_{TT}^TR_{0f} & D_{RT}^T\bar{G}^{-1}D_{RT}-R_{0c} \end{bmatrix}_{2n\times 2n}\sqrt{G_0} \qquad (7a)$$

$$B_1=B_2\begin{bmatrix} 0 & 0 \\ -D_{RT}^T\bar{G}^{-1} & -D_{RT}^T\bar{G}^{-1}D_{RR}R_c^{-1} \end{bmatrix}_{2n\times d} \qquad (7b)$$

$$B_2=K^T\sqrt{R_0}\begin{bmatrix} U_l & -D_{TT} \\ D_{TT}^TR_{0f} & D_{RT}^T\bar{G}^{-1}D_{RT}+R_{0c} \end{bmatrix}_{2n\times 2n}^{-1} \qquad (7c)$$

where U_l is a unit matrix of order l.

An alternative form of the state equation can be obtained if the maximum number of transmission line edges are in the forest $(D_{RT}=0)$.

For the state equation to exist the inverted matrix in (7) must be nonsingular. This condition is satisfied if the matrices R_{0c}, R_{0f}, R_c and G_f are either positive or negative definite. This in turn is satisfied after the application of the E-shift and I-shift theorems to eliminate degenerate voltage and current sources in the sections. Degenerate edges that terminate transmission lines will not affect the above condition.

It should be noted that the state equation in (6) and (7) is basically the scattering representation of the sections through all the transmission line ports, normalized to the characteristic impedances of the lines. The topological form of the matrices A, B_1, and B_2 are similar to those of the scattering matrix representation in the frequency domain of networks containing lumped elements [2].

When the z-transform is applied to (6) we obtain the state equation in the z-domain

$$\begin{bmatrix} b_f(z) \\ b_c(z) \end{bmatrix}=\left[zU_{2n}-A\right]^{-1}\left[B_1\begin{bmatrix} I_R(z) \\ E_R(z) \end{bmatrix}+B_2\begin{bmatrix} I_T(z) \\ E_T(z) \end{bmatrix}\right] \qquad (8)$$

where $z=e^{sT}=e^{(\sigma+j\omega)T}=\Sigma+j\Omega$.

For transmission line networks the number of eigenvalues, in the s-domain, of the state matrix A are infinite. However, for commensurate networks the eigenvalues are periodic with respect to ωT with a period of $\omega T=\pi$ and there are $2n$ eigenvalues in every 2π period. This means that the total number of eigenvalues in the z-plane is $2n$ and that they occur in conjugate pairs.

IV. The Output Equation

In general the output vector $y(t)$ will have some elements in the forest and some in the coforest. The output equation is given by

$$\begin{bmatrix} y_f(t) \\ y_c(t) \end{bmatrix}=C\begin{bmatrix} b_f(t) \\ b_c(t) \end{bmatrix}+D_1\begin{bmatrix} I_R(t) \\ E_R(t) \end{bmatrix}+D_2\begin{bmatrix} I_T(t) \\ E_T(t) \end{bmatrix}. \qquad (9)$$

We shall give the results for four different output vectors which will cover every possible voltage or current in the network. In every case the full output equation need not be used, but the equations related to the desired output can be extracted and solved. All the following results are for $D_{TR}=0$ but similar results can be obtained for $D_{RT}=0$.

A. $y(t)=[v_{Tf}(t) \vdots i_{Tc}(t)]^T$

This gives all the transmission line voltages in the forest and all the transmission line currents in the coforest. The matrices C, D_1, and D_2 are given the symbols C_T, D_{1T}, and D_{2T} and are given by

$$C_T=D_{2T}\begin{bmatrix} 2\sqrt{G_{0f}} & 0 \\ 0 & -2\sqrt{R_{0c}} \end{bmatrix}_{2n\times 2n} \tag{10a}$$

$$D_{1T}=D_{2T}\begin{bmatrix} 0 & 0 \\ -D_{RT}^T\bar{G}^{-1} & -D_{RT}^T\bar{G}^{-1}D_{RR}R_c^{-1} \end{bmatrix}_{2n\times d} \tag{10b}$$

$$D_{2T}=\begin{bmatrix} G_{0f} & -D_{TT} \\ D_{TT}^T & R_{0c}+D_{RT}^T\bar{G}^{-1}D_{RT} \end{bmatrix}^{-1}_{2n\times 2n}. \tag{10c}$$

B. $y(t)=[i_{Tf}(t) \vdots v_{Tc}(t)]^T$

This gives all the transmission line currents in the forest and all the transmission line voltages in the coforest. The matrices C, D_1, and D_2 are given the symbols $\bar{C}_T$, $\bar{D}_{1T}$, and $\bar{D}_{2T}$ and are given by

$$\bar{C}_T=\bar{D}_{2T}\begin{bmatrix} 0 & -2D_{TT}\sqrt{G_{0c}} \\ -2D_{TT}^T\sqrt{R_{0f}} & 2D_{RT}^T\bar{G}^{-1}D_{RT}\sqrt{G_{0c}} \end{bmatrix}_{2n\times 2n} \tag{11a}$$

$$\bar{D}_{1T}=\bar{D}_{2T}\begin{bmatrix} 0 & 0 \\ -D_{RT}^T\bar{G}^{-1} & -D_{RT}^T\bar{G}^{-1}D_{RR}R_c^{-1} \end{bmatrix}_{2n\times d} \tag{11b}$$

$$\bar{D}_{2T}=\begin{bmatrix} U_l & -D_{TT}G_{0c} \\ D_{TT}^TR_{0f} & U_m+D_{RT}^T\bar{G}^{-1}D_{RT}G_{0c} \end{bmatrix}^{-1}_{2n\times 2n}. \tag{11c}$$

C. $y(t)=[v_{Rf}(t) \vdots i_{Rc}(t)]^T$

This gives all the resistive element voltages in the forest and currents in the coforest. The matrices C, D_1, and D_2 are given the symbols C_R, D_{1R}, D_{2R} and are given by

$$C_R=MC_T \tag{12a}$$

$$D_{1R}=\begin{bmatrix} G_f & -D_{RR} \\ D_{RR}^T & R_c \end{bmatrix}^{-1}_{d\times d}+MD_{1T} \tag{12b}$$

$$D_{2R}=MD_{2T} \tag{12c}$$

where C_T, D_{1T}, and D_{2T} are given by (10) and

$$M=\begin{bmatrix} 0 & \bar{G}^{-1}D_{RT} \\ 0 & -R_c^{-1}D_{RR}^T\bar{G}^{-1}D_{RT} \end{bmatrix}_{d\times 2n}. \tag{13}$$

D. $y(t)=[i_{Rf}(t) \vdots v_{Rc}(t)]^T$

The matrices C, D_1, and D_2 for this case are given the symbols $\bar{C}_R$, $\bar{D}_{1R}$, and $\bar{D}_{2R}$ and are obtained by premultiplying C_R, D_{1R}, and D_{2R} by

$$\begin{bmatrix} G_f & 0 \\ 0 & R_c \end{bmatrix}_{d\times d}.$$

From (8) and (9) the transfer equation in the z-domain can be found

$$\begin{bmatrix} y_f(z) \\ y_c(z) \end{bmatrix}=\left[C[zU_{2n}-A]^{-1}B_1+D_1\right]\begin{bmatrix} I_R(z) \\ E_R(z) \end{bmatrix} +\left[C[zU_{2n}-A]^{-1}B_2+D_2\right]\begin{bmatrix} I_T(z) \\ E_T(z) \end{bmatrix}. \tag{14}$$

The transfer function between any chosen input and output can then be calculated.

V. Noncommensurate Networks

The topological forms of the matrices A, B_1, B_2, C, D_1, and D_2 given in Sections III and IV are valid for commensurate and noncommensurate networks. In the noncommensurate case the delays on the lines could be all different and the time advanced state vector in (6) should be replaced by

$$\left[b_{f1}(t+\alpha_1T)\cdots b_{fl}(t+\alpha_lT) \vdots b_{c(l+1)} \cdot(t+\alpha_{l+1}T)\cdots b_{c(l+m)}(t+\alpha_{l+m}T)\right]^T$$

where $\alpha_1\cdots\alpha_{2n}$ are the ratios between the delays on each line and the normalized or "standard" delay T. The normalization could also be made with respect to any one of the lines in the network.

It is always possible to choose the delay T such that the ratio between the highest and the lowest α is not greater than 2. If a line has too great a delay it is divided into two lines. Such a choice will enable a meaningful comparison to exist between a noncommensurate network and a commensurate prototype network obtained by setting all the α ratios to 1. The relation between the noncommensurate network and its commensurate prototype is explained in Section VI.

To obtain the transfer equation for noncommensurate networks the matrix $[zU_{2n}-A]$ in (14) is replaced by

$$\left[\mathrm{diag}(z^{\alpha 1},z^{\alpha 2},\cdots,z^{\alpha 2n})-A\right]. \tag{15}$$

The characteristic function is the determinant of the matrix in (15) and the natural frequencies are the roots of this function. The characteristic function is no longer rational and the number of its roots are no longer finite. In this case each root has a different period and an overall period cannot be defined. It is also clear that the eigenvalues of the state matrix A no longer represent the natural frequencies.

VI. Properties of Transmission Line Networks

In this section, we shall discuss some properties of transmission line networks and in particular the order of the transfer function and the degree of freedom for these networks.

We first define the following quantities for commensurate networks.

The fundamental delay T_F is the delay on each commensurate line.

The fundamental state and output equations are the equations written with respect to the fundamental delay T_F. The matrices A_F, B_F, C_F, and D_F in these equations will be called fundamental matrices.

The fundamental natural frequencies are the nonzero eigenvalues of the fundamental state matrix A_F.

The fundamental angular period θ_F is the angle in the z-plane where all the fundamental natural frequencies occur.

The order of the network n_F is the number of fundamental natural frequencies or the rank of the fundamental state matrix A_F.

The degree of freedom d_F is the number of fundamental natural frequencies that can be simultaneously adjusted given the set of unknown network parameters. Since the natural frequencies are in conjugate pairs only half the fundamental natural frequencies are independent, but since each has a real and imaginary part the degree of freedom is equal to the total number.

For lumped networks the order of the network is the same as the degree of freedom which is also the order of the state matrix or the number of finite natural frequencies including those at $s=0$ [3]–[5]. This is consistent with the definition of the order of the network n_F given above since the zero natural frequencies in the z-plane are transformed to $\sigma=-\infty$ in the s-plane.

It is not essential to write the state equation with respect to T_F, but a "commensurate" delay T could be chosen such that $T=T_F/i$ where i is a positive integer. The total number of natural frequencies in the z-plane will be in_F. In this case the fundamental angular period $\theta_F=2\pi/i$ can be identified by the periodic nature of the natural frequencies.

A. Normal Commensurate Networks

In normal commensurate networks the signals arriving at any edge have delays which are even multiples of T the one way delay on each commensurate line [6]. Any output $y(t)$ can be expressed in terms of the input $u(t)$

$$y(t)=\sum_{n=0}^{\infty} a_n u(t-2nT) \tag{16}$$

and the fundamental delay T_F is $2T$.

If the state equation is written in terms of T then $\theta_F=\pi$. If the state matrix is nonsingular then $n_F=n=$ the number of transmission lines in the network, while the total number of natural frequencies in the z-plane is $2n$. The unknowns are the characteristic resistances of the lines and the degree of freedom or the number of restrictions that can be imposed on the transfer function is n.

As an example we consider a normal commensurate network consisting of r cascaded lines and s shunt stubs as shown in Fig. 3. When the state equation is written with respect to the one way delay T on each line, there will be $2n$ natural frequencies in the z-plane where $n=r+s$ is the total number of lines. The natural frequencies will be symmetrical about $\omega T=\pm\pi/2$, the fundamental period θ_F is π and $n_F=d_F=n$. There are also r transmission zeros at $z=0$ and $z=\infty$ and s transmission zeros at $z=j$ and $z=-j$.

One of the most common methods of designing filters of the type described above is to use the Kuroda identities to calculate the characteristic resistances of the cascaded lines. In this case the state matrix A becomes nonsingular and its rank is reduced by r. Since we have imposed restrictions on the characteristic resistances the degree of freedom is also reduced by r and we have $n_F=d_F=s$. In the z-plane $2r$ natural frequencies will be at $z=0$ and the resulting pole-zero pattern is shown in Fig. 3.

B. Nonnormal Commensurate Networks

Networks of this type have their fundamental delay equal to the one way delay on each line and if A_F is nonsingular then $n_F=2n$. An example is shown in Fig. 4. In the z-plane no symmetry exists apart from that about the real axis. However, since there are still only n unknown network parameters the degree of freedom $d_F=n$.

C. Noncommensurate Networks

When the ratio of the largest α to the smallest α is less than 2 then a noncommensurate network can be considered as a perturbation of a prototype commensurate network obtained by setting all the α ratios to 1. The order of the noncommensurate network will be defined as the order of the prototype network and all the delays are written in terms of the commensurate delay. Each natural frequency of a noncommensurate network will have a different period, thus in any θ_F period no symmetry exists and each fundamental natural frequency can be independently adjusted. Since the number of unknowns are now the line lengths as well as their characteristic resistances then $d_F=2n$.

Table I gives a comparison between the types of networks discussed above, in each case the total number of transmission lines is n.

VII. The Analysis Program

The analysis program is based on the direct application of (14) to calculate any transfer function. The input to the program are the parameters l, m, a, and b and the matrices K, D, R, and R_0. Also required is a directive to

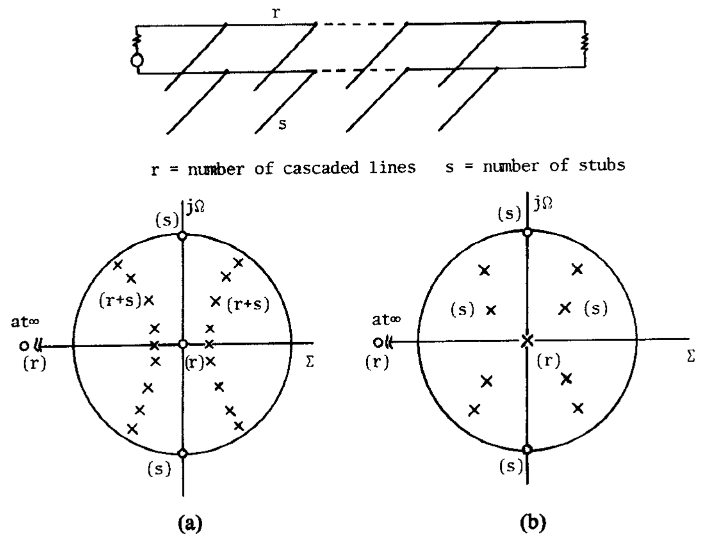


Fig. 3. (a) Example of pole-zero locations in the z-plane of a normal network. (b) Pole-zero locations of networks designed by using Kuroda's identities.

indicate which transfer functions are to be calculated and the range of frequencies of interest. In the case of noncommensurate networks, the vector α of the delay ratios is also part of the input requirements.

The outputs from the program are the natural frequencies and the zeros of transmission in the z, s, and λ domains ($\lambda = \tanh sT$) for each transfer function and a plot of the magnitude and phase of the transfer functions with respect to ω.

The program starts by forming the matrices D_{TT}, D_{TT}^T, D_{RT}, D_{RT}^T, D_{RR}, D_{RR}^T, G_f, R_c, R_{0f}, and R_{0c} where they exist. This step is followed by forming the matrices A, B_1, B_2, C, D_1, and D_2. For every transfer function the vectors B and D are formed, where

$$B = B_{1,2}\left[\frac{I_{R,T}(z)}{E_{R,T}(z)}\right] \text{ and } D = D_{1,2}\left[\frac{I_{R,T}(z)}{E_{R,T}(z)}\right]. \quad (17)$$

In (17) the current or voltage source for the required transfer function is set to 1 with all the other sources set to 0.

The above steps are common between commensurate and noncommensurate networks. They are followed by the calculation of the natural frequencies, the transmission zeros and the magnitudes and phase of the transfer functions. These steps are different for the two types of networks.

A. Commensurate Networks

The natural frequencies of commensurate networks are calculated from the eigenvalues of the state matrix A. The zeros of the transfer function are obtained from $C[zU-A]^{-1}B+D=0$. The expansion of $[zU-A]^{-1}$ is achieved by the Faddeeva algorithm [7]. The magnitude and phase of the transfer function $C[zU-A]^{-1}B+D$ are then calculated for the required values of frequency ω.

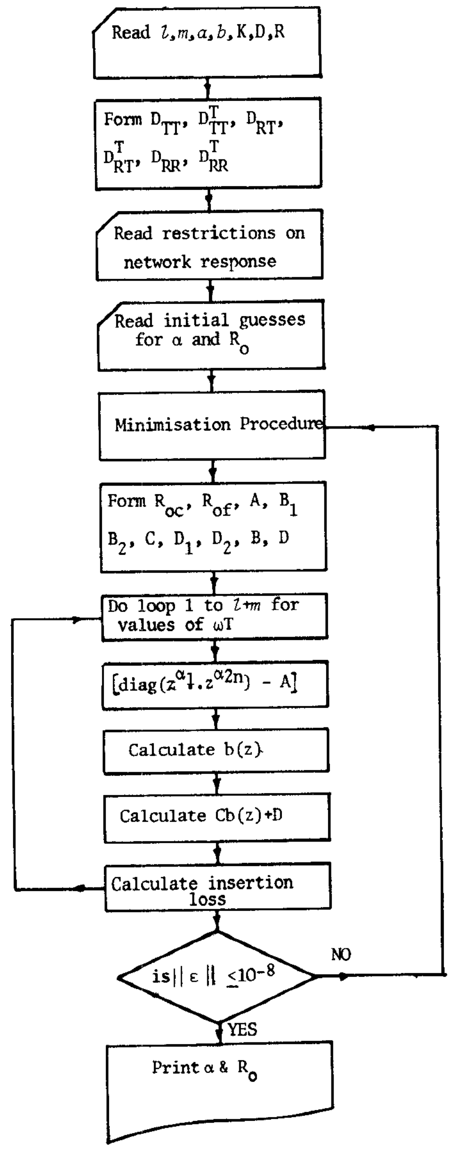


Fig. 4. Example of a nonnormal network.

TABLE I
THE ORDER AND THE DEGREE OF FREEDOM OF VARIOUS TYPES OF NETWORKS CONTAINING N TRANSMISSION LINES.

	n_F	d_F
Normal commensurate	n	n
Networks designed by Kuroda identities with r cascaded and s shunt lines	s	s
Nonnormal commensurate	$2n$	n
Normal noncommensurate	n	$2n$
Nonnormal noncommensurate	$2n$	$2n$

B. Noncommensurate Networks

The natural frequencies for these networks are obtained by equating the determinant of the matrix in (15) to 0. This requires an iteration procedure for minimising the function which in turn requires a set of initial guesses for the location of the roots. Also, since the number of these roots are infinite even in the z-plane the frequency limits must be defined. The initial guesses used are the eigenvalues of the state matrix which is equivalent to assuming that the noncommensurate network is obtained by perturbing the commensurate network. If the perturbation in the values of the delay ratios is high then the process could be done in more than one step starting with the commensurate case and introducing smaller perturbations in the elements of the vector α.

The Faddeeva algorithm can no longer be used to expand the inverse of (15) except to obtain the initial guesses for the zeros by setting $\alpha_i = 1$. When the initial guesses are known an iterative procedure is used to solve the equation $Cb(z) + D = 0$ where $b(z)$ is the state vector which is calculated from the state equation for every value of z and D is given by (18).

The magnitude and phase of the transfer function $Cb(z) + D$ are then calculated at all the required frequencies.

VIII. The Synthesis Programs

All the synthesis programs developed use an iteration procedure to minimize a set of conditions and calculate the set of unknown network parameters. Two main programs were developed, the first is suitable for commensurate networks when the natural frequencies can be specified and the second is a generalized program in which specifications could be made directly on the transfer function and both commensurate and noncommensurate networks could be synthesized.

The iterative optimization procedure is based on the generalized least squares method. A function $f(x)$ of n independent variables x is optimized to meet n conditions such that the norm of the error vector is minimum. The elements of the error vector are the differences between the desired conditions and the achieved response. The errors could be weighted if more accuracy is desired for some of the conditions.

The conditions on the function form a set of non-linear equations.

$$F_i(x) = 0 \qquad \text{where } i = 1, \cdots, n. \tag{18}$$

An initial guess x_0 for the vector of the unknowns $x_1 \cdots x_n$ is supplied to the procedure and a new vector x_1 is calculated from

$$x_1 = x_0 - \left[\mathcal{J}^T(x_0)\mathcal{J}(x_0) + \alpha U_n \right]^{-1} \mathcal{J}^T(x_0) F(x_0) \tag{19}$$

where $\mathcal{J}$ is the Jacobian matrix of the system of functions $F(x_0)$ and α is a constant which if set to zero we obtain the Newton-Gauss method and if set to a large value we obtain the steepest descent method.

The error vector ϵ is given by the values of the functions $F_i(x_1)$. If the norm of the error vector is more than a specified accuracy the vector x_0 in (19) is replaced by x_1 and a new vector x_2 is obtained. The process is repeated until the desired accuracy is achieved.

A. Commensurate Networks with Specified Natural Frequencies

A large number of practical networks fall into this category. Special attention was given to the case of equiripple responses and a subroutine was developed to calculate the location of the natural frequencies for an equiripple response given the locations of the transmission zeros. The procedure is based on making the approximation in the λ-plane when the order of the filter, the ripple factor and the cutoff frequency are known. The mathematical steps for this method are well known [8] and need not be repeated here.

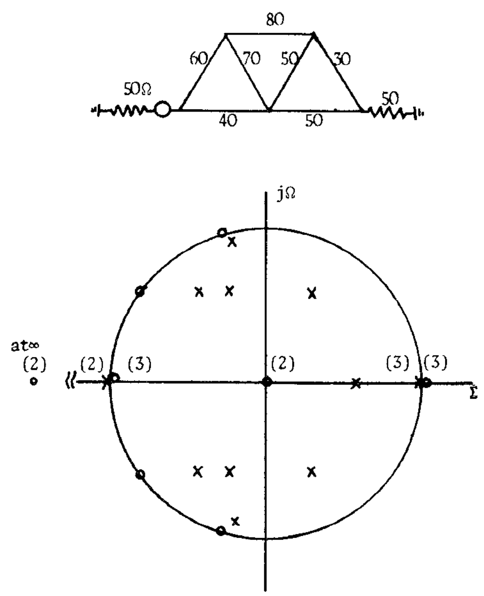


Fig. 5. Flowchart of generalized synthesis program.

From the natural frequencies the coefficient of the characteristic polynomial are calculated and the minimisation procedure matches the actual coefficients to the required ones. The elements of the error vector are the differences between the actual and required coefficients and the procedure terminates when the norm of the error vector is less than 10^{-8}.

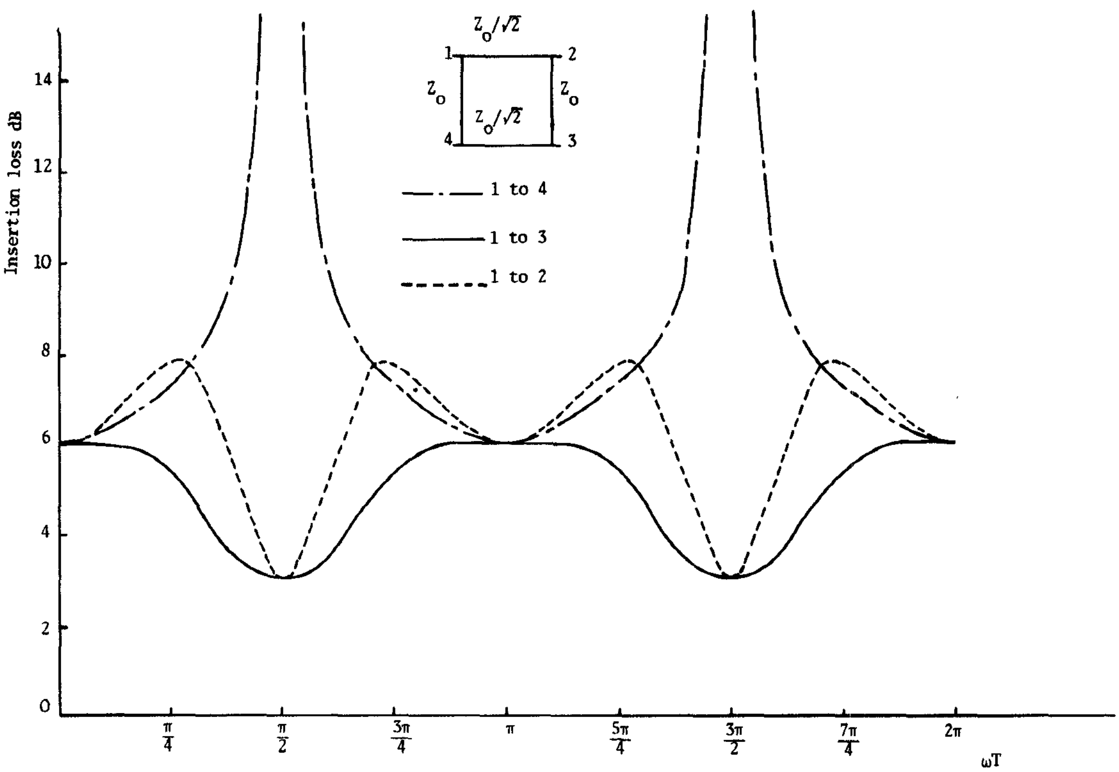
B. The Generalized Synthesis Program

In this program a generalized set of restrictions are specified and the corresponding set of functions are formulated. A flow chart of the program is shown in Fig. 5.

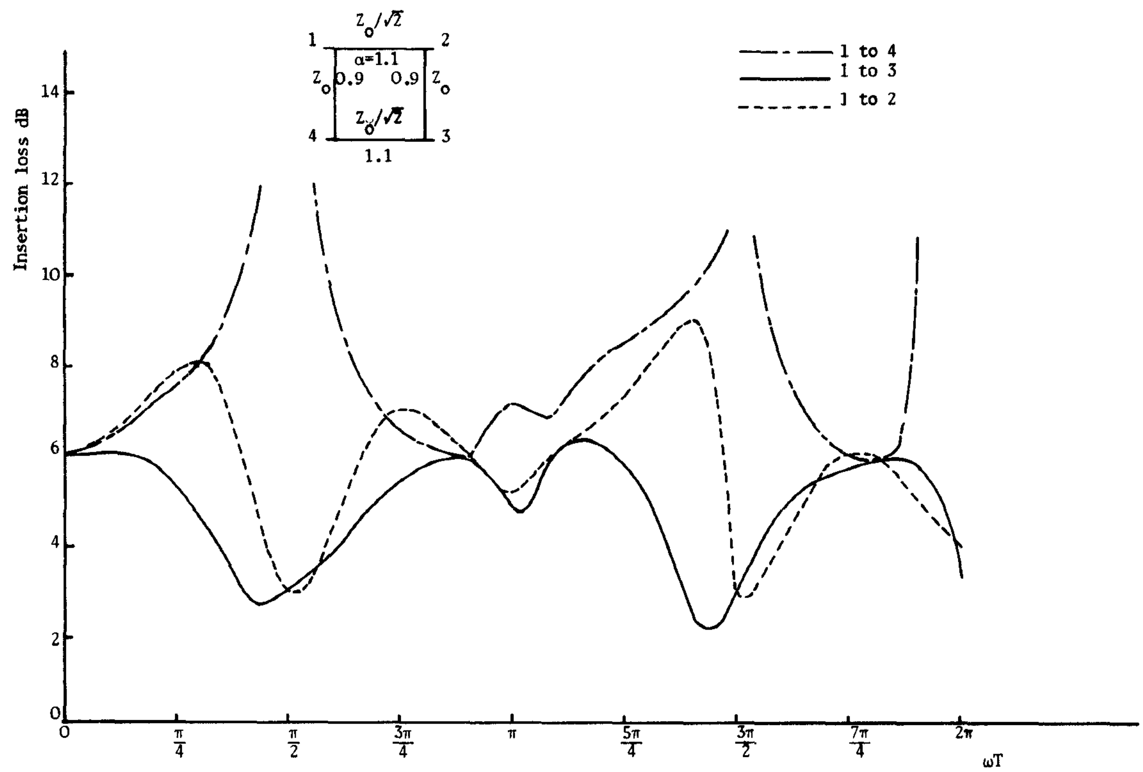
After the matrices D_{TT}, D_{TT}^T, D_{RT}, D_{RT}^T, D_{RR}, D_{RR}^T, G_f, and R_c are formed, the initial guesses for the vector α and the matrix R_0 are used to calculate the matrices A, B_1, B_2, C, D_1, and D_2 and the initial values for the specified restrictions. An error vector is formed and a minimisation procedure is used to reduce the norm of the error vector to less than 10^{-8}. The norm of the error vector gives the sum of the squares of the differences between the desired restrictions and the achieved response.

IX. Examples

Some examples are given below to illustrate the use of the analysis and synthesis programs. All the programs are written in Algol and run on an ICL 4130 machine. This is a slow machine by present day standards and the computer times given below should be judged accordingly.



(a)



(b)

Fig. 6. (a) Response of commensurate hybrid (Example I). (b) Response of noncommensurate hybrid (Example I).

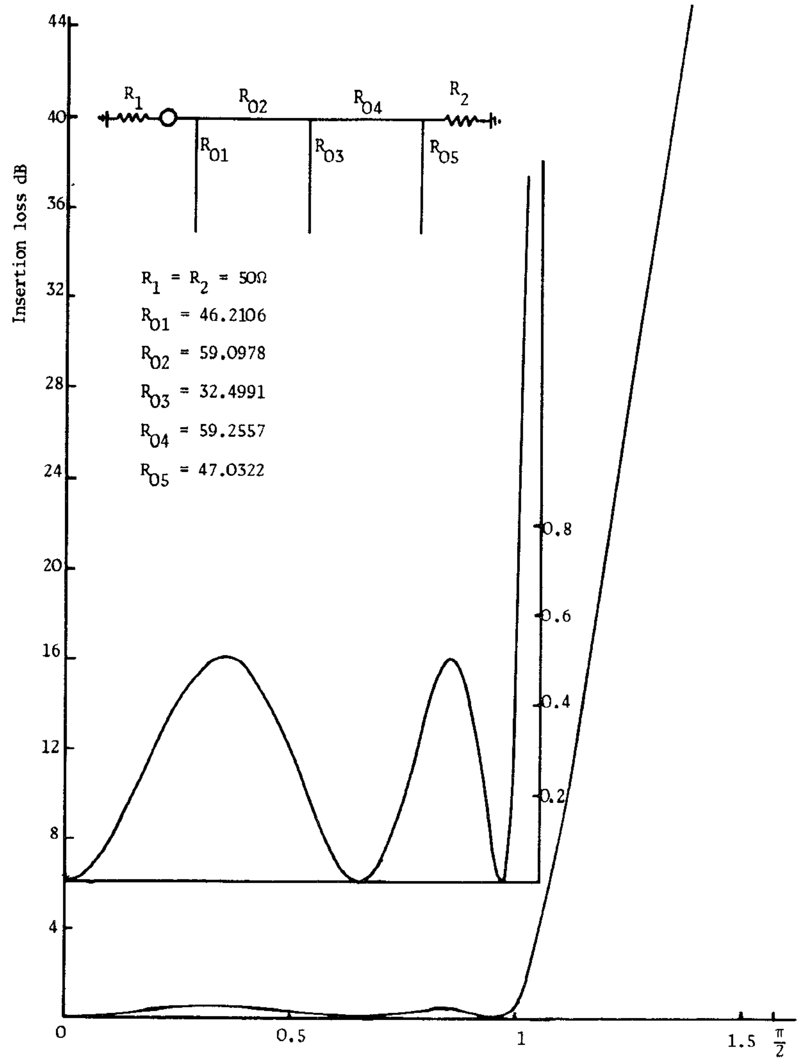


Fig. 7. Nonredundant fifth-order Chebyshev filter with 0.5-dB ripple (Example II).

For all the synthesis examples, the solutions given correspond to the norm of the error vector being less than 10^{-8}.

Example I: In this example the analysis program was used to study the effect of that changes of the line lengths would have on the performance of a branch hybrid network. The commensurate case was solved first and the response is shown in Fig. 6(a). When the line lengths are perturbed then the noncommensurate case was analyzed and the response is shown in Fig. 6(b). The computer times were 2 min and 26 min, respectively.

Example II: This is a synthesis problem for the fifth-order commensurate network shown in Fig. 7. A nonredundant equiripple response is required with 0.5-dB ripple and a cutoff angle ωT of 1. First the required values of the natural frequencies in the z-plane were obtained given that the network has three transmission zeros at $+j$ and at $-j$ and two transmission zeros at 0 and at ∞. The characteristic impedances of the lines were calculated and they are given in Fig. 7 together with the network response.

The total computer time was 5 min and the number of iterations was 36.

Example III: In this example ten restrictions were applied on the response of a fifth-order noncommensurate network. The insertion loss 'a' was specified at 10 frequencies and the full list of restrictions was as follows:

a in dB	3.85	0.3	0.5	0.45	0.4	1.8	3.5	10	39	41.2
at ωT	2.094	2.304	2.775	3.508	3.979	4.189	4.293	4.356	4.497	4.618

In Fig. 8 the values of the line lengths and characteristic impedances are given and a plot of the response is shown with the specification points marked.

The computer time was 15 min and the number of iterations was 15.

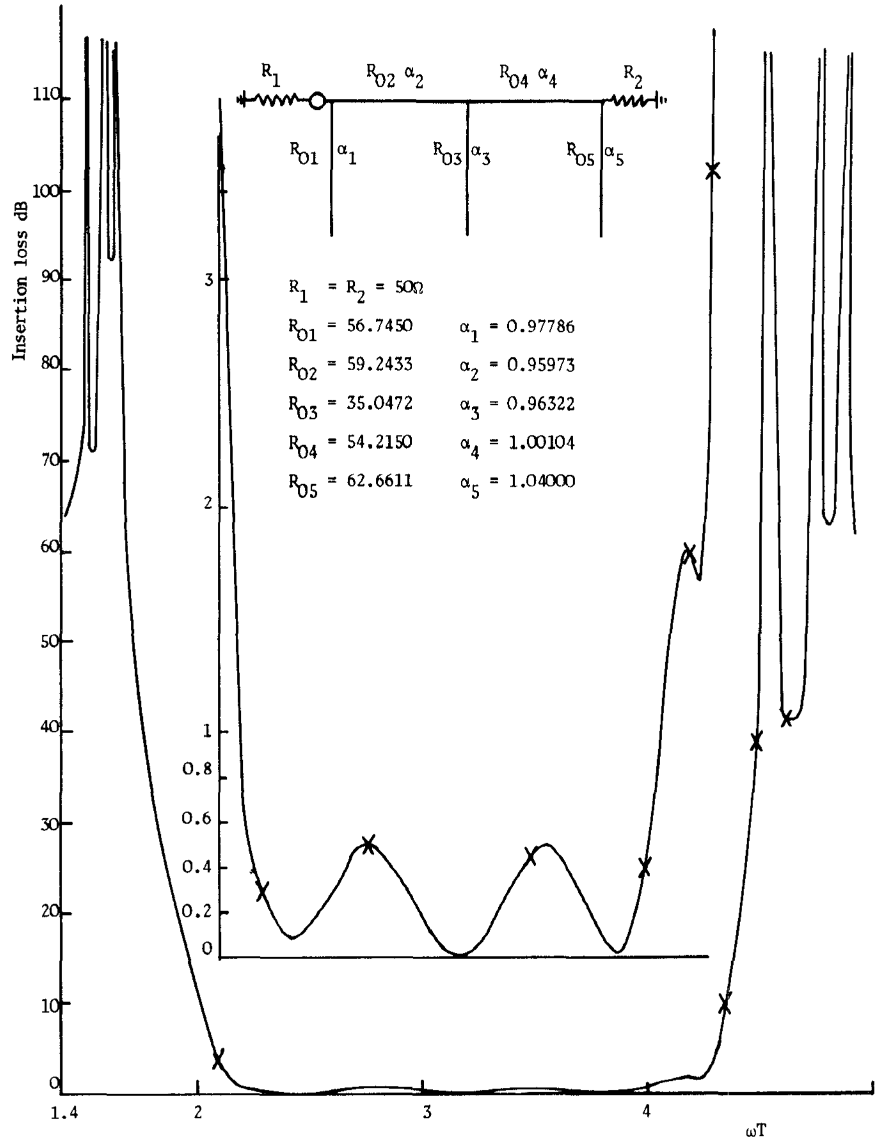


Fig. 8. Realization of ten restrictions on the response of a fifth-order filter (Example III).

Example IV: In this example a noncommensurate third order network was designed to meet the following six conditions:

1) a zero of transmission at $\omega T = 1.5$;
2) a zero of transmission at $\omega T = 1.6$;
3) a minimum insertion loss of 50 dB in the range of $\omega T = 1.5$ and 1.6;
4) and 5) both maxima in the passband should be 0.5 dB (2 conditions);
6) the lower cutoff angle ωT is 2.2.

The results are shown in Fig. 9 and the specified restrictions are marked on the response curve.

The maxima were obtained by numerical differentiation.

The computer time was 50 min. and the number of iterations was 14. The long computer time in this case was due to having to differentiate the transfer function several times in every iteration.

X. Conclusions

The combination of topological methods and state-space methods gives a general and powerful technique for the analysis and synthesis of transmission line methods. No restrictions in the topology are required and both commensurate and noncommensurate methods can be handled. In the developed computer programs the restrictions can be either specified directly on the network response or by specifying the locations of the natural frequencies.

The basic methods could be extended to cover a much wider variety of problems some of which are listed below:

1) time-domain analysis and synthesis;
2) coupled transmission line networks;
3) nonuniform lines;
4) networks containing both lumped and distributed elements;
5) parasitic effects in transmission line networks;
6) the analysis and synthesis of digital filters.

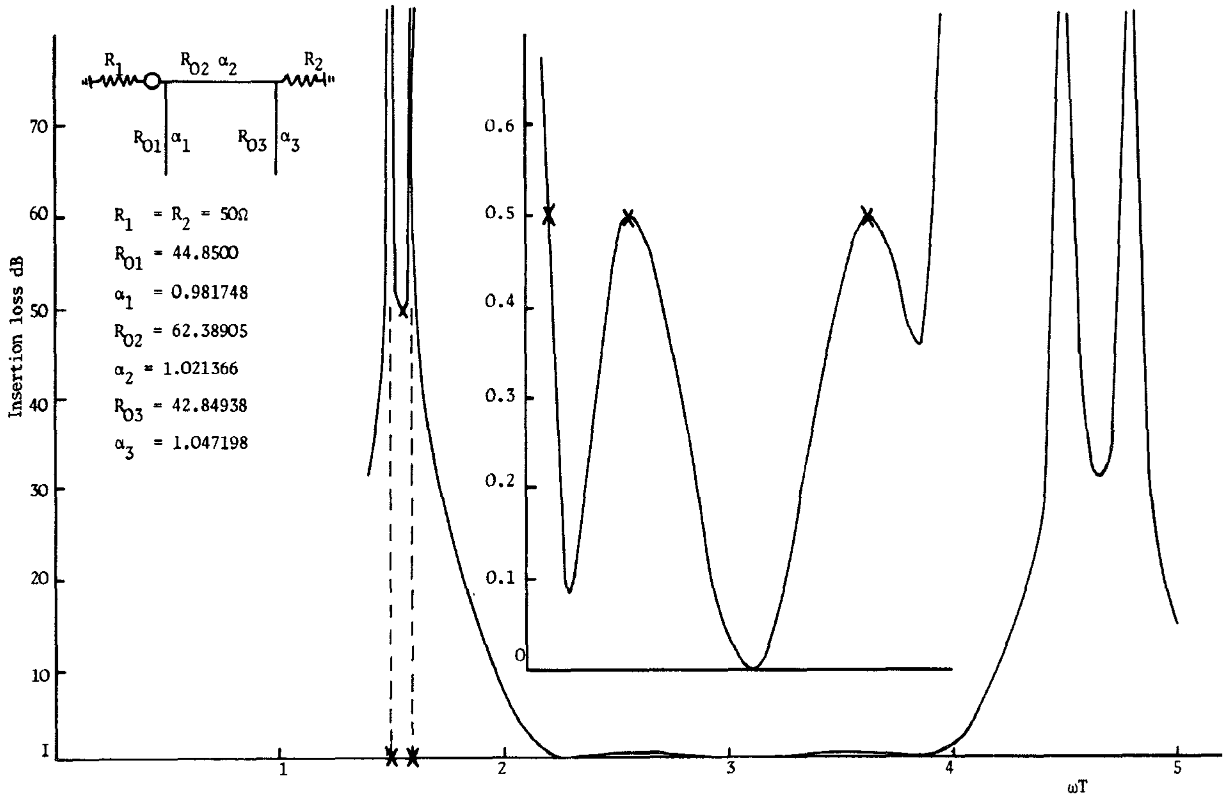


Fig. 9. Realization of six restrictions on the response of a third-order filter (Example IV).

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